NUMERICAL SOLUTION OF AN INVERSE HEAT-CONDUCTION BOUNDARY PROBLEM

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We consider a computational algorithm for the solution of an inverse heat-conduction boundary-value problem based on the replacement of one of the boundary conditions by a nonlocal condition. We present examples of our calculations for a one-dimensional model problem.

Among the inverse problems of heat transfer [1], much attention has been given to inverse boundary-value problems of heat conduction. In problems of this kind thermal measurements made on one portion of the boundary of a specimen are to be used to recover the thermal loading on another portion of the boundary inaccessible to measurement. In the practical one-dimensional case (as our specimen we take a thin homogeneous rod with a thermally insulated lateral surface) the problem consists in solving the heat-conduction equation with a known initial temperature and a given temperature and thermal flow at one end. Such a problem belongs to the class of problems conditionally correct in the Tikhonov sense [1, 2]. To solve it approximately various computational algorithms may be applied [1, 3, 4] based on the use of integral boundary conditions and the schematic involved in A. N. Tikhonov's method of regularization.

In the present paper we employ an approach involving a perturbation of the boundary conditions. In connection with stationary inverse boundary problems of heat conduction, modelled by means of a Cauchy problem for elliptic equations, such an approach was considered in [5]. To numerically solve the resulting nonlocal parabolic problem we apply the usual difference methods [6]. This allows for easy transition to nonlinear multidimensional problems. Difference methods were applied earlier in [1, 7] to obtain an approximate solution of inverse boundary problems. The numerical calculations we present make it possible, in a certain sense, to indicate the working range of the methods when used to solve specified applied problems.

Perturbed Problem. Let it be required to determine u(x, t) from the conditions

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ 0<x<l,\ 0<t<T; \tag{1}$$

$$u(x,\ 0)=u_0(x),\ 0<x<l; \tag{2}$$

$$u(0,\ t)=\varphi(t),\ 0<t<T; \tag{3}$$

$$\frac{\partial u}{\partial x}(0,\ t)=0,\ 0<t<T, \tag{4}$$

wherein it is assumed that all the thermophysical characteristics of the specimen are constant, the left end (x = 0) is thermally insulated, and at this end a temperature measurement is made. We note that the temperature (or thermal flux) can be measured at an arbitrary second point x^* ($0 < x^* < \ell$) of the specimen.

We seek an approximate solution $u_\alpha(x, t)$ of the heat-conduction equation

$$\frac{\partial u_\alpha}{\partial t}=\frac{\partial^2 u_\alpha}{\partial x^2},\ 0<x<l,\ 0<t<T, \tag{5}$$

with the initial condition

$$u_\alpha(x,\ 0)=u_0(x),\ 0<x<l, \tag{6}$$

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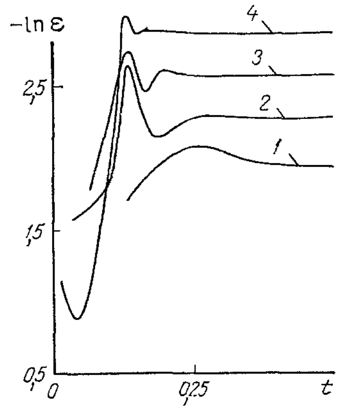

Fig. 1. Dependence of ε(t) on the network employed with $\alpha = 10^{-14}$ and $\sigma = 1$: Curves labeled 1, 2, 3, and 4 for networks of 9 × 5, 17 × 9, 33 × 17, and 65 × 33, respectively.

and the boundary condition

$$\frac{\partial u_\alpha}{\partial x}(0, t) = 0, \ 0 < t < T. \tag{7}$$

Instead of the boundary condition (3), we consider a nonlocal condition connecting the solution at x = 0 and x = ℓ:

$$u_\alpha(0, t) + \alpha u_\alpha(l, t) = \varphi(t). \tag{8}$$

In the initial problem with errors the functions $u_0(x)$ and $\varphi(t)$ are specified; the value of the numerical parameter $\alpha > 0$ in the nonlocal condition (8) is determined by averaging these errors.

In contrast to the quasiinversion method used in [7], it is not the initial equation in Eqs. (5)-(8) that is perturbed, but only the boundary condition. Perturbation of the boundary (initial) conditions and not the equation itself is, in many problems of mathematical physics, a more natural procedure since the boundary (initial) conditions are often known approximately. A nonlocal perturbation of the initial condition in a retrospective inverse problem of heat conduction (a problem with reverse time) was considered in [8]. This approach was described in [5] in connection with stationary inverse boundary problems of heat conduction.

A Difference Problem and Its Numerical Solution. We consider now the problem of numerically solving the nonlocal problem (5)-(8). We introduce a uniform network $\omega = \omega_h \times \omega_\tau$, where

$$\omega_h = \{x_i = ih, \ h > 0, \ i = 1, 2, \ldots, M-1, \ Mh = l\},$$
$$\omega_\tau = \{t_j = j\tau, \ \tau > 0, \ j = 1, 2, \ldots, N-1, \ N\tau = T\}.$$

We put the differential problem (5)-(8) into correspondence with a difference problem. We approximate Eq. (5) on ω by an implicit difference scheme [6] of the following kind:

$$\frac{y_i^{j+1} - y_i^j}{\tau} = \sigma \Lambda y_i^{j+1} + (1-\sigma)\Lambda y_i^j, \tag{9}$$

where the usual notation of difference schemes is employed:

$$y_i^j = y(x_i, t_j), \ \Lambda y_i^j = y_{\bar{x}x}^j = \frac{y_{i+1}^j - 2y_i^j + y_{i-1}^j}{h^2}.$$

Approximating the initial condition (6), we have

$$y_i^0 = u_0(x_i), \ x_i \in \omega_h. \tag{10}$$

We approximate the boundary condition (7) for the solution [6] with second order in space

$$\sigma y_x^{j+1} + (1-\sigma) y_x^j = \frac{h}{2} y_t, \ i = 0. \tag{11}$$

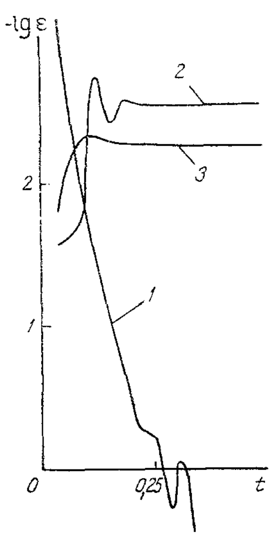

Fig. 2

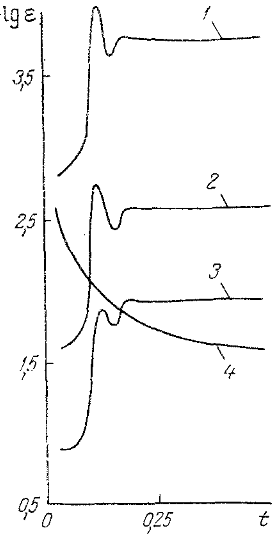

Fig. 3

Fig. 2. Relative accuracy ε for different schemes ((33 × 17), $\alpha = 10^{-14}$): Curves 1, 2, and 3 correspond to σ values of 0.5, 1, and 1.5, respectively.

Fig. 3. Accuracy of the purely implicit scheme (σ = 1) versus the exact solution when the 33 × 17 network is used: curves 1, 2, 3, and 4 correspond to k values of 8, 4, 4/3, and 2.

In Eq. (11) we have used

$$y_x^j = \frac{y_{i+1}^j - y_i^j}{h}, \quad y_t = \frac{y_i^{j+1} - y_i^j}{\tau}.$$

The nonlocal condition (8) leads to the relation

$$y_0^{j+1} + \alpha y_M^{j+1} = \varphi(t_{j+1}), \ t_j \in \omega_\tau. \tag{12}$$

To solve the difference problem (9)-(12) at the (j + 1)-st time layer we apply a specific algorithm involving a three-point segment (see [5, 9]) in which the nonlocal condition (12) is taken into account.

Examples of Calculations. The computational algorithm proposed was tested on numerous examples. We present the results of our calculations, made on the BÉSM-6 computer, for the model example (1)-(4), which has the exact solution

$$(x,\ t) = \cos\left(\frac{\pi}{k}x\right)\exp\left(-\frac{\pi^2}{k^2}t\right) \tag{13}$$

for various values of the numerical parameter k. In the data presented below from our computational experiment the input information is given exactly. Errors are only due to discretization of the differential problem and to round-off errors. An estimate of the effect of errors in prescribing the input information ($u_0(x)$ and $\varphi(t)$) on the precision of the approximate solution requires a special investigation. Preliminary handling of the input information is of great significance here. Calculations were carried out for sufficiently small values of the parameter α, so that a perturbation of a boundary condition has not significance: $\alpha \max |u_\alpha(\ell,\ t)| \leq 10^{-10}$.

In Fig. 1, for Eq. (13) with k = 4, T = 0.5, and ℓ = 1 (Fourier number Fo = 1), we show, for $\alpha = 10^{-14}$, how the accuracy ε of the difference solution depends on the network size employed. The relative accuracy ε is determined from the formula

$$\varepsilon = \varepsilon(t_j) = \frac{\max\limits_{x\in\omega_h} |u(x_i,\ t_j) - y_i^j|}{\max\limits_{x\in\omega_h} |y_i^j|}.$$

We applied a purely implicit (σ = 1) scheme given by relations (9)-(12). For the given sample, exact and approximate values of the thermal flux for two different networks, (33 × 17) and (65 × 33), for T = 0.5 and T = 0.25, are shown in Table 1. The dependence of ε on α is as

TABLE 1. Accuracy of Thermal Flux Calculations at Certain Points

$q=\frac{\partial u}{\partial x}\cdot 10$	Exact solution	Network dimensions	
		(33×17)	(65×33)
$T=0,5$	0,5655	0,5710	0,5682
	1,1256	1,1364	1,1310
	1,6748	1,6907	1,6828
	2,2079	2,2287	2,2183
	2,7198	2,7449	2,7323
	3,2054	3,2345	3,2199
	3,6602	3,6926	3,6764
$T=0,25$	0,6598	0,6662	0,6630
	1,3133	1,3259	1,3196
	1,9541	1,9727	1,9634
	2,5761	2,6003	2,5882
	3,1732	3,2026	3,1879
	3,7399	3,7738	3,7568
	4,2705	4,3082	4,2893

follows. For large α (more precisely, for large $\alpha \max |u_\alpha(\ell, t)|$) the error of the difference solution is large; subsequently, ε decreases and, up to some sufficiently small α, achieves a characteristic plateau. For small α the error again increases; this growth is determined by the effect of the approximation and round-off errors. In Fig. 2 we show the results of our calculations when different values of σ are employed in the relations (9)-(12). Naturally, the scheme with $\sigma = 0.5$, having second order of approximation also with respect to the time, yields higher accuracy for small t. The scheme with $\sigma = 1$ has better asymptotic characteristics [6], which, in the given case, become apparent with the great accuracy at large times. It should be noted that the advantage of the symmetric scheme is again manifested in the various solutions. In particular, this scheme gives a very large accuracy on the solutions (13) for $k = 2$. The dependence of ε on t for the purely implicit scheme is shown in Fig. 3 for the different solutions. We note, particularly, that for $k = 2$ and $\alpha = 1$ the perturbation in the boundary condition (the term $\alpha u_\alpha(\ell, t)$) is small since $u_\alpha(\ell, t) \approx 0$.

NOTATION

$u(x, t)$, temperature; ℓ, rod length; T, time interval; u_α, approximate solution; α, regularization parameter; ω, difference network; $u_0(x)$, initial temperature of the rod; $\varphi(t)$, rod temperature at end $x = 0$; y_i^j, network solution; Λ, second difference; σ, weight of difference scheme; ε, relative error.

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